Math 275D Lecture 11 Notes

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1 Distribution of The Last Zero in $[0, 1]$ and Martingale Properties of Brownian Motion

1.1 First time to exceed a value

Proposition 1.1. *Let* $T_a = \inf\{t > 0 : B_t \geq a\}$. *Then*

$$\mathbb{P}(T_a < 1) = 2\mathbb{P}(B_1 > a).$$

Proof. Last time, we said

$$\mathbb{P}(B_1 > a) = \mathbb{P}(B_1 > a \mid T_a < 1) \cdot \mathbb{P}(T_a < 1).$$

So we want to show that $\mathbb{P}(B_1 > a \mid T_a < 1) = \frac{1}{2}$. We have

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{B_1 > B_{T_a}\}} \mid \mathcal{F}_a] &= \mathbb{E}[\mathbb{1}_{\{B_{1-T_a} > B_0\}} \circ \theta_{T_a} \mid \mathcal{F}_a] \\ &= \mathbb{E}_{B_{T_a}}[\mathbb{1}_{\{B_{1-T_a} > B_0\}}] \\ &= \mathbb{E}_a[\mathbb{1}_{\{B_{1-T_a} > a\}}].\end{aligned}$$

This is $1/2$ if $T_a < 1$. □

Corollary 1.1. *Let* Φ *denote the CDF of the standard Gaussian distribution. Then*

$$\mathbb{P}(T_a < 1) = 2(1 - \Phi(a)).$$

1.2 Distribution of the last zero in $[0, 1]$

We've shown that $\inf\{t > 0 : B_t = 0\} = 0$ a.s. What is the distribution of the last zero in $[0, 1]$? Let $A = \sup\{0 \leq t \leq 1 : B_t = 0\}$. Then

$$\mathbb{P}(A \leq t) = \int p_t(0, y) \cdot \mathbb{P}(\text{no zeros between } t \text{ and } 1 \mid B(t) = y)\, dy.$$

By shifting the Brownian motion by t, the probability in the integrand is

$$\mathbb{P}_0(T_{-y} > 1-t) = 2(1-\Phi(y)).$$

After solving the integral, we get

$$\mathbb{P}(A \le t) = \frac{2}{\pi}\arcsin(t).$$

Another related question: Let $a = \sup_{t\in[0,1]} B(t)$, and let $B(T_s) = a$. What is the distribution of T_s?

1.3 Martingale properties of Brownian motion

Definition 1.1. A random function $(X_t)_{t\ge 0}$ is a **martingale** (with respect to the filtration $\mathcal{F}_t$) if for all $t > s$, $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$.

Equivalently, the condition is $\mathbb{E}[X_t - X_s \mid \mathcal{F}_s] = 0$.

Proposition 1.2. *$X_t = B_t^2 - t$ is a martingale.*

Proof. First, $X_t - X_s = B_t^2 - B_s^2 - (t-s)$. Then $B_t = B_s + Y$, where $Y \perp B_s$ and $Y \sim N(0, t-s)$. So

$$\mathbb{E}[X_t - X_s \mid \mathcal{F}_s] = \mathbb{E}[Y^2 + 2Y - (t-s) \mid \mathcal{F}_s] = \mathbb{E}[Y^2] + 2\,\mathbb{E}[Y] - (t-s) = 0. \qquad \square$$

Proposition 1.3. *Let $T = \inf\{t > 0 : B(t) \in \{a, b\}\}$. Then $\mathbb{E}[T] = -ab$.*

Proof. Since $X_t = B_t^2 - t$ is a martingale,

$$\mathbb{E}[B_T^2] - \mathbb{E}[T] = \mathbb{E}[X_T] = \mathbb{E}[X_0] = 0.$$

To find $\mathbb{E}[B_T^2]$, we have $\mathbb{E}[B_T^2] = \mathbb{P}(B_T = a)a^2 + \mathbb{P}(B_T = b)b^2$. Since B_t is a martingle, $\mathbb{E}[B_T] = 0$. So we can calculate

$$\mathbb{P}(B_T = a) = -\frac{b}{a}(1 - \mathbb{P}(B_T = a)) \implies \mathbb{P}(B_T = a) = \frac{b}{b-a}.$$

So we get

$$\mathbb{E}[T] = -\frac{a^2 b}{b-a} + \frac{b^2 a}{b-a} = -ab. \qquad \square$$

What if we want to find $\mathbb{E}[T^2]$? We can use another martingale with a B_t^4 term. Next time, we will talk about how to figure out such martingales involving Brownian motion.